A Comprehensive Reliability Analysis of Redundant Systems

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The general effect of redundancy on system reliability is analyzed and discussed. Three types of redundancy are described—active redundancy, standby redundancy, and active-standby redundancy—and equations are developed for each type of redundancy. The most significant contribution is the development of a Poisson-binomial probability distribution function (PDF) which applies to active-standby redundancy and is the general case of the Poisson and binomial PDF's. Special attention is given to the most general case (active-standby redundancy with start and switching; i.e., a modified Poisson-binomial PDF). The modified Poisson-binomial is applicable to systems where 1) the operating units have a constant failure rate characteristic of the operating mode, 2) the standby units have a different failure rate (also constant) characteristic of the standby mode, 3) all units are subject to failure when started, and 4) the standby units are also subject to failure when switched into the system. A rapid and accurate approximation technique for analyzing active-standby redundancy is also presented.

Nomenclature

$\bar{m}$	=	mean number of units operating for the mission time (binomial PDF)
M	=	required number of operating units
N	=	number of standby units available at the start of the mission
PDF	=	probability distribution function
Q_{st}	=	$1 - R_{st}$ = start failure probability
Q_{sw}	=	$1 - R_{sw}$ = switching failure probability
Q_{stsw}	=	$1 - R_{stsw}$ = start or switching failure probability
Q_u	=	$1 - R_u$ = unit failure probability
R_{M+N}	=	mission reliability with M operating units and N initial standby units
R_{st}, R_{sw}	=	start and switching success probabilities, respectively
R_{stsw}	=	$R_{st} \cdot R_{sw}$
R_u	=	unit reliability = $e^{-\lambda_0 t}$
$R_u{}^M$	=	$(1 - \beta)^{\alpha}$ = reliability of M operating units
$S_n^{(m)}$	=	Stirling numbers of the first kind
t	=	mission time, hr
α	=	$M\lambda_0/\lambda_s$
β	=	$1 - e^{-\lambda_s t}$
$\Gamma(x)$	=	gamma function of x
ϵ	=	the smaller of M and N
θ	=	$M\lambda_0 t + (\bar{m} - M)\lambda_s t$
λ_0, λ_s	=	operating and standby mode failure rates, respectively, hr^{-1}

Introduction

ONE method of achieving high system reliability for long, manned space missions is through the use of high levels of redundancy, which falls into three distinct categories: active redundancy, standby redundancy, and a combination of both which will be referred to as active-standby redundancy. Active redundancy can be analyzed by using the binomial probability distribution function (PDF); standby redundancy can be analyzed by using the Poisson PDF. Active-standby redundancy can now be analyzed by using the Poisson-binomial PDF developed in this paper. The latter is also presented in a modified and even more general form by including the effects of imperfect subsystem start and switching probabilities. The final, general equation presented allows the accurate calculation of mission reliability for M active units and N standby units for which the standby failure rate is different than the operating failure rate; all units have a certain success probability for starting, and the N standby units also have a success probability for switching. An accurate approximation to the Poisson-binomial PDF is also provided for rapid determination of the reliabilities of redundant systems. The approximation is obtained by determining a mean number of operating units and using this mean value in the Poisson PDF.

Analysis and Discussion

The method employed in the development of the following reliability equations is the integration of the appropriate probability density function. These equations apply only to units subjected to random failures and which, therefore, have constant failure rates. The failure rates may be different for the operating and standby modes, but they are constant during a mode. The exponential probability density function is $f(t) = \lambda_0 e^{-\lambda_0 t}$. The equation for unit reliability is obtained by integration of this function[1] to obtain the corresponding exponential PDF, $R_u = e^{-\lambda_0 t}$, which is the probability that a unit with failure rate λ_0 will fail after time t.

In the case of complex systems it is necessary to 1) define all of the combinations of success (success paths), 2) obtain the appropriate density function for each combination, 3) integrate each density function to obtain the expression for the probability of success due to that combination, and 4) sum these probability expressions to obtain the total reliability equation. The success paths for a system consisting of one operating unit and two standby (redundant) units, a 1 + 2 system, are shown in Fig. 1. For example, path

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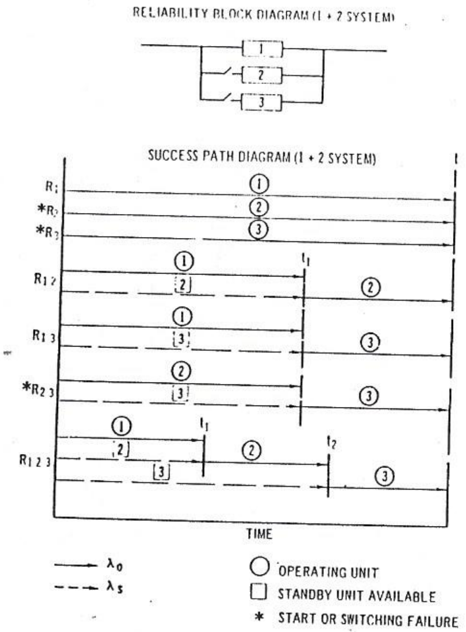

Fig. 1 Reliability diagrams for a 1 + 2 system.

R_{12} applies to the case where unit 1 starts the mission and fails at time t_1; unit 2 survives in the standby mode until t_1, is started and switched into the system at t_1, and operates from t_1 to the mission time t. Only the relevant success paths are shown (e.g., the survival of units 2 and 3 in the standby mode for path R_1 is irrelevant), and a standby unit indicated as being available requires the survival of that unit in the standby mode until it is required to operate.

The system reliability equations herein are multivariate[2] PDF's and are functions of mission time (t) and the number of operating (M) and standby (N) units. The use of these functions as multivariate PDF's is proper for reliability engineering since the substitution of the exponential PDF into the Poisson and binomial distributions produces equations (and hence PDF's) which are functions of M, N, and t. One of the requirements of a PDF is that the sum of the probabilities for all possible conditions is unity. In reliability engineering this sum is divided into the two distinct states of system success and system failure. The reliability then becomes a truncated PDF consisting of only the terms which represent success. The complete PDF does exist, however, and may be completed by the addition of the terms representing system failure.

The following sections will treat active redundancy, standby redundancy, and active-standby redundancy—all with perfect start and switching. The effect of imperfect start and switching will then be examined.

Active Redundancy

Active redundancy is the simplest and most common form of redundancy methods. For active redundancy 1) all units ($M + N$) are operating at the start of the mission (or the failure rates of the standby and operating units are equal, $\lambda_s = \lambda_0$), and 2) the binomial (or negative binomial) PDF is employed for system reliability calculations. The binomial PDF is basic to the field of reliability.[3] It is

$$R_{M+N} = \sum_{n=0}^{N} \binom{M+N}{n} [1 - e^{-\lambda_0 t}]^n [e^{-\lambda_0 t}]^{M+N-n} \quad (1)$$

Active redundancy can also be represented by the negative binomial PDF, which is

$$R_{M+N} = \sum_{n=0}^{N} \binom{n+M-1}{n} [1 - e^{-\lambda_0 t}]^n [e^{-\lambda_0 t}]^M \quad (1a)$$

A brief discussion of the negative binomial PDF is presented in the Appendix.

Equations (1) represent the reliability of a system with $M + N$ units at the start of the mission where at least M units are required to be operating at the end of the mission.

Standby Redundancy

Standby redundancy provides the greatest reliability improvement of all redundancy methods. For standby redundancy there are no failures in the standby mode ($\lambda_s = 0$), and the Poisson PDF is employed for system reliability call culations. Although the equations for calculating the reliability of a system employing standby redundancy are wel-established,[1,4] the equation for a system with one operating unit and one standby unit (1 + 1) will be developed to demonstrate the validity of the analytical method. A "1 + 1" system provides two success paths, R_1 and R_{12} (Fig. 1). The reliability of the first path $R_1 = R_u = e^{-\lambda_0 t}$. The second path is described by the following differential reliability:

dR_{12} = Probability [unit 1 operates from $t = 0$ to t_1 and fails at t_1 within the interval dt_1] × Probability [unit 2 operates from t_1 to t]

Since t_1 can vary from time 0 to the mission time t, the differential reliability is integrated as follows:

$$R_{12} = \int_0^t [e^{-\lambda_0 t_1}\lambda_0\, dt_1][e^{-\lambda_0(t-t_1)}] = \lambda_0 t e^{-\lambda_0 t}$$

and

$$R_{1+1} = R_1 + R_{12} = e^{-\lambda_0 t}(1 + \lambda_0 t) \quad (2)$$

By continuing in this manner, the Poisson PDF for M active units and N standby units may be obtained and is

$$R_{M+N} = e^{-M\lambda_0 t} \sum_{n=0}^{N} \frac{(M\lambda_0 t)^n}{n!} \quad (3)$$

which represents the reliability as a function of time of a system with M operating units and N initial standby units.

Active-Standby Redundancy

Active-standby redundancy is a compromise between active and standby redundancy. It represents a more realistic appraisal of the actual situation when units are standing by in a nonoperating mode, since it allows these units to also experience random failures; hence, pure standby redundancy equations do not apply. Since the standby units are normally subjected to a less severe environment than the operating units, the failure rate of the standby units is normally less than that of the operating units; therefore, active redundancy equations also do not apply. This type of redundancy should employ the Poisson-binomial PDF for reliability calculations; this PDF is developed below.

The reliability equation for a "1 + 1" active-standby system can be obtained by redeveloping the term R_{12} and adding it to R_1. Thus

dR_{12} = Probability [unit 1 operates from $t = 0$ to t_1 and fails at t_1 within the interval dt_1] × Probability [unit 2 survives in the standby mode from $t = 0$ to t_1] × Probability [unit 2 operates from t_1 to t]

Inserting the appropriate probability density function and integrating gives

$$R_{12} = \int_0^t [e^{-\lambda_0 t_1}\cdot \lambda_0 dt_1][e^{-\lambda_s t_1}][e^{-\lambda_0(t-t_1)}]$$

Thus,

$$R_{12} = e^{-\lambda_0 t}(\lambda_0/\lambda_s)(1 - e^{-\lambda_s t})$$

and since the path R_1 is unaffected

$$R_{1+1} = e^{-\lambda_0 t}[1 + (\lambda_0/\lambda_s)(1 - e^{-\lambda_s t})] \tag{4}$$

Equation (4) is the Poisson-binomial PDF for a "1 + 1" active-standby redundant system and has been previously developed.[1,5] This equation degenerates to the Poisson PDF for $\lambda_s = 0$‡ and to the negative binomial PDF for $\lambda_s = \lambda_0$.

The reliability for a "1 + 2" active-standby redundant system can be obtained by determining R_{13} and R_{123} (Fig. 1) and adding these terms to Eq. (4). The result is

$$R_{1+2} = e^{-\lambda_0 t}\left[1 + \frac{\lambda_0}{\lambda_s}(1 - e^{-\lambda_s t}) + \left(\frac{\lambda_0}{\lambda_s} + 1\right)\frac{\lambda_0}{\lambda_s}\frac{(1 - e^{-\lambda_s t})^2}{2}\right] \tag{5}$$

Since each additional standby unit complicates the Poisson-binomial PDF, the following substitutions are made to facilitate the algebra:

$$\alpha = M(\lambda_0/\lambda_s) \qquad \text{(for the "1 + 2" system, } M = 1)$$

$$\beta = 1 - e^{-\lambda_s t} \qquad R_u = e^{-\lambda_0 t}$$

Equation (5) can then be written as

$$R_{1+2} = R_u\,[1 + \alpha\beta + (\alpha + 1)\alpha(\beta^2/2)] \tag{5a}$$

The foregoing procedure may be continued with additional standby units in an effort to establish a general equation. When this is accomplished, the result for a "1 + N" redundant system is

$$R_{1+N} = \sum_{n=0}^{N} \binom{n + \alpha - 1}{n} \beta^n (1 - \beta)^\alpha \tag{6}$$

where it may be noted that $(1 - \beta)^\alpha = e^{-\lambda_0 t} = R_u$.

In Eq. (6), α is greater than 0, but α is not necessarily an integer. The binomial coefficients $\binom{x}{r}$ are defined as being applicable "for all values of x and all positive integers r."[2] For example,

$$\binom{\alpha + 2}{3} = \frac{(\alpha + 2)!}{3!(\alpha - 1)!} = \frac{(\alpha + 2)(\alpha + 1)\alpha}{3!}$$

for all values of α. For $\alpha = 0.5$, for example,

$$\binom{0.5 + 2}{3} = \frac{2.5(1.5)(0.5)}{6} = 0.3125$$

The Poisson-binomial coefficient in Eq. (6) can also be expressed in terms of the gamma functions as

$$\binom{n + \alpha - 1}{n} = \frac{\Gamma(n + \alpha)}{n!\,\Gamma(\alpha)}$$

This may also be evaluated for the foregoing example ($n = 3$, $\alpha = 0.5$) as

$$\binom{n + \alpha - 1}{n} = \frac{\Gamma(3.5)}{3!\Gamma(0.5)} = \frac{3.323}{6(1.772)} = 0.3125$$

‡ Applying l'Hospital's rule as follows:

$$\lambda_0 \lim_{\lambda_s \to 0} \frac{1 - e^{-\lambda_s t}}{\lambda_s} = \lambda_0 \lim_{\lambda_s \to 0} t e^{-\lambda_s t} = \lambda_0 t$$

If a different algebraic combination of the terms is employed, the following equivalent equation is obtained:

$$R_{1+N} = R_u \sum_{m=0}^{N} \alpha^m \sum_{n=m}^{N} (-1)^{n-m} S_n^{(m)} \cdot \frac{\beta^n}{n!} \tag{6a}$$

where $S_n^{(m)}$ represents the Stirling numbers of the first kind and the term $(-1)^{n-m} S_n^{(m)}$ is the number of permutations of n things which have exactly m cycles.[6,7] For example, the success paths R_{1245}, R_{1345}, and R_{1235} each represent four things, units 2, 3, 4, and 5, having three cycles. (A cycle in this sense is the replacement of an operating unit with the next available standby unit.)

Equations (6) represent the Poisson-binomial PDF for a "1 + N" system with active-standby redundancy. Equations (6) can be modified slightly to obtain the general Poisson-binomial PDF for an "M + N" system. This is accomplished by replacing R_u by R_u^M and using the more general form of $\alpha = M(\lambda_0/\lambda_s)$; i.e., $(1 - \beta)^\alpha = R_u^M$. The general equation for an "M + N" system is

$$R_{M+N} = \sum_{n=0}^{N} \binom{n + \alpha - 1}{n} \beta^n (1 - \beta)^\alpha \tag{7}$$

or

$$R_{M+N} = R_u^M \sum_{m=0}^{N} \alpha^m \sum_{n=m}^{N} (-1)^{n-m} S_n^{(m)} \frac{\beta^n}{n!} \tag{7a}$$

or

$$R_{M+N} = \sum_{n=0}^{N} \binom{\alpha + N}{n} \beta^n (1 - \beta)^{\alpha + N - n} \tag{7b}$$

each of which represents the reliability as a function of time of a system with M operating units (λ_0) and N initial standby units (λ_s). Equations (7) represent the Poisson-binomial PDF for an "M + N" system with active-standby redundancy. This PDF is the more general form of the Poisson and bi-nomial PDF's and provides a cohesive element in understanding the relationship between active, standby, and active-standby redundancy. It degenerates to the negative binomial or binomial PDF for $\lambda_s = \lambda_0$ and to the Poisson PDF for $\lambda_s = 0$. (This may be demonstrated for $\lambda_s = 0$ by the successive application of l'Hospital's rule.) Equation (7) provides a simplified equation that uses noninteger, negative binomial-type coefficients to expedite calculations. Equation (7a) presents the coefficients of the terms in a form that can be studied using combinatorial analysis. Equation (7b) is similar to Eq. (7), but it uses noninteger, binomial-type coefficients.

Figure 2 shows the manner in which the Poisson-binomial PDF covers the range between the Poisson and binomial PDF's for a "2 + 3" system as λ_s varies from 0 to λ_0 for three values of $\lambda_0 t$ (referring to the three solid curves for which $R_{st} = 1.0$; the remaining curves are discussed later).

It should be noted, however, that the magnitude of the standby failure rate is not bounded by the operating failure rate; i.e., λ_s can be greater than λ_0. This is shown in Fig. 3 for $\lambda_0 t = 0.3$; $M = 1$ and 2; and $N = 1, 2$, and 3. The reliability curve starts at a value corresponding to the Poisson PDF for small values of λ_s/λ_0, passes through the binomial PDF value for $\lambda_s/\lambda_0 = 1.0$, and approaches $e^{-M\lambda_0 t}$ for very large values of λ_s/λ_0 where the standby units fail too rapidly to be useful. The latter results ($\lambda_s > \lambda_0$) would apply in a case where, although it would be more reliable to operate all units in parallel, other system constraints necessitate standby redundancy at the higher standby failure rate. The system reliability may still be improved significantly over the reliability of a system with no redundancy.

In addition to Eqs. (7), a Poisson-binomial approximation has been developed for quickly estimating system reliability with active-standby redundancy. The approximation can be employed using standard tables of the Poisson PDF, thus

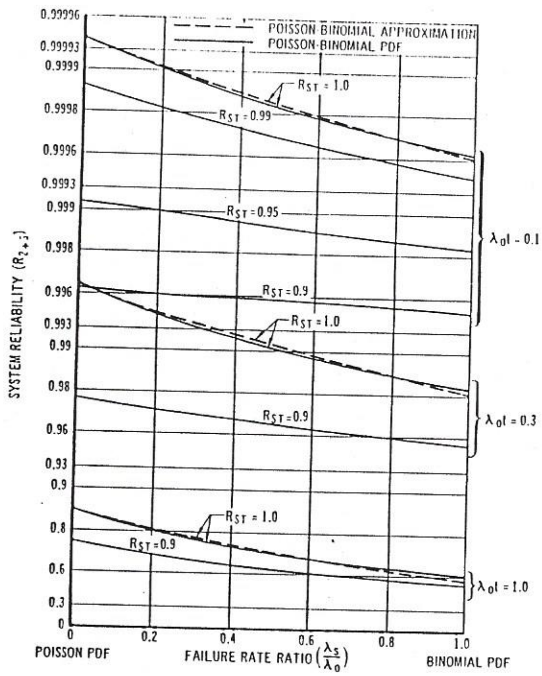

Fig. 2 Reliability of a 2 + 3 active-standby redundant system.

allowing rapid estimates of reliability and extensive system parametric studies.

The Poisson-binomial approximation is based on the hypothesis that the binomial PDF is a special case of the Poisson PDF. In the binomial PDF, the number of units subjected to the failure rate is not constant since "$M + N$" units are available at the start of the mission and at least M units are operating at the end of a successful mission. Thus, in order to use the Poisson PDF as an approximation to the binomial PDF, it is necessary to determine the mean number of units ($\bar{m}$) operating for the entire mission time. The Poisson PDF is employed as follows as an approximation of an active-standby redundant system:

$$R_{M+N} \cong e^{-\theta} \sum_{n=0}^{N} \frac{\theta^n}{n!} \tag{8}$$

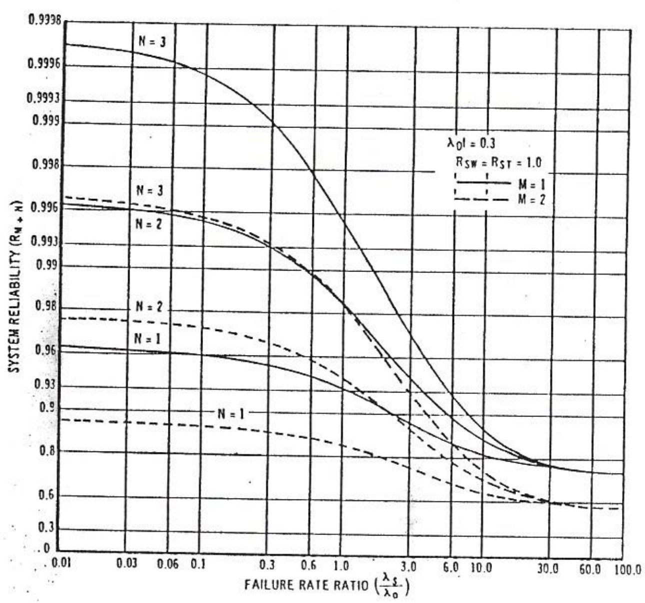

Fig. 3 Reliability of an $M + N$ active-standby redundant system.

where

$$\theta = M\lambda_0 t + (\bar{m} - M)\lambda_s t \tag{9}$$

and

$$\bar{m} = \left[\prod_{j=0}^{N} (M + j)\right]^{1/(N+1)} \tag{10}$$

The term $\bar{m}$ is the mean number of units operating for the mission time; the factor $(\bar{m} - M)$ results since, of the $\bar{m}$ units operating for the entire mission time, M of these are subjected to the operating failure rate λ_0 whereas the remaining units are subjected to the standby failure rate λ_s. The general value of $\bar{m}$ is obtained by setting $\lambda_s = \lambda_0$ and equating the resulting expression, Eq. (8), to the negative binomial PDF. Thus,

$$e^{-\bar{m}\lambda_0 t} \sum_{n=0}^{N} \frac{(\bar{m}\lambda_0 t)^n}{n!} = e^{-M\lambda_0 t} \sum_{x=0}^{N} \binom{x + M - 1}{x} \times (1 - e^{-\lambda_0 t})^x \tag{11}$$

which is the defining equation for $\bar{m}$. It is apparent that $\bar{m}$ is a function of M, N, and $\lambda_0 t$. The equation presented previously for $\bar{m}$, Eq. (10), was derived by utilizing the Taylor Series representation of Eq. (11) and letting $\lambda_0 t \to 0$. The value of $\bar{m}$ is a generalized geometric mean and might appropriately be called a binomial mean. Thus, for $\lambda_0 t \to 0$, $\bar{m}$ is independent of $\lambda_0 t$, as indicated in Eq. (10). In fact, as is shown by the dashed lines in Fig. 2, the value of $\bar{m}$ as obtained from Eq. (10) results in a good approximation to the Poisson-binomial PDF even for values of $\lambda_0 t$ approaching unity, thus indicating a weak functional relationship between $\lambda_0 t$ and $\bar{m}$. The Poisson-binomial approximation is seen to be quite accurate over its range of definition ($0 \leq \lambda_s \leq \lambda_0$) and is easier to use than Eq. (7) in that, once the equivalent failure rate function θ is computed using Eqs. (9) and (10), the tables of the cumulative Poisson distribution may be used to obtain the system reliability. For the "2 + 3" system presented in Fig. 2, $\bar{m} = 3.31$ from Eq. (10).

Unit Start and Switching Effects

A modified Poisson-binomial PDF that includes the effects of imperfect unit start and switching is also of practical interest. The equation is derived for the general case of active-standby redundancy ($\lambda_s \neq \lambda_0$). However, the results will also be useful, in the appropriate limiting forms, for active redundancy ($\lambda_s = \lambda_0$) or standby redundancy ($\lambda_s = 0$). The difference between the start and switching reliabilities

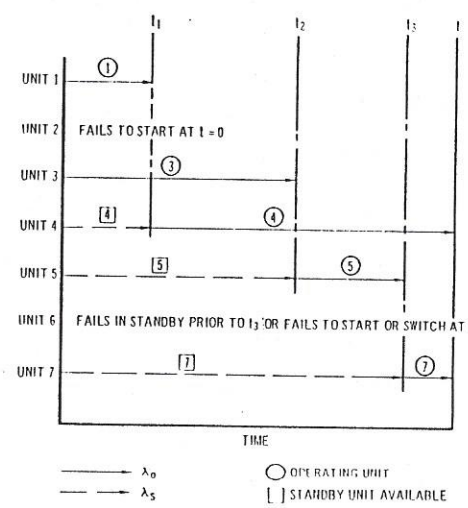

Fig. 4 Reliability diagram of success path $R_{14,357}$.

(or probabilities, since each is assumed to be independent of time) is as follows: the initial M active units are already switched into the system and must only be started (R_{st}) whereas the standby units, when called upon, must start and be switched (R_{stsw}) into the system. Since a similar analytical approach is employed to obtain the final equation, only one success path will be presented employing the start and switching probabilities. A complex path will better illustrate the probability logic involved; thus, one path of a "2 + 5" system will be examined.

The success path $R_{14,357}$ is shown in Fig. 4 and is interpreted thusly: at the beginning of the mission, unit 1 is started, unit 2 fails to start, and unit 3 is started and switched into the system; at t_1, unit 1 fails and unit 4 is started, switched in, and completes the mission; at t_2, $t_2 \geq t_1$, unit 3 fails and unit 5 is started and switched in; at t_3, unit 5 fails, unit 6 is not available (it has failed either in standby, fails to start, or fails to switch), unit 7 is started, switched in, and completes the mission. Hence, letting $Q_{st} = 1 - R_{st}$

$$R_{14,357} = \int_{t_3=0}^{t}\int_{t_2=0}^{t_3}\int_{t_1=0}^{t_2} [R_{st}e^{-\lambda_0 t_1}\lambda_0 dt_1][e^{-\lambda_s t_1}] \times [R_{stsw}e^{-\lambda_0(t-t_1)}][Q_{st}][R_{stsw}e^{-\lambda_0 t_2}\lambda_0 dt_2][e^{-\lambda_s t_2}] \times [R_{stsw}e^{-\lambda_0(t_3-t_2)}\,\lambda_0 dt_3][1 - R_{stsw}e^{-\lambda_s t_3}][e^{-\lambda_s t_3}][R_{stsw}e^{-\lambda_0(t-t_3)}]$$

Integrating and letting $Q_{stsw} = 1 - R_{stsw}$

$$R_{14,357} = e^{-2\lambda_0 t}R_{st}Q_{st}\left(2\frac{\lambda_0}{\lambda_s}\right)^3 \times \frac{1}{16}\left[\frac{(1-e^{-\lambda_s t})^3}{3}R_{stsw}{}^4Q_{stsw} + \frac{(1-e^{-\lambda_s t})^4}{4}R_{stsw}{}^5\right]$$

Substituting α and β as before

$$R_{14,357} = e^{-2\lambda_0 t}\cdot R_{st}Q_{st}\cdot(\alpha^3/16) \times [(\beta^3/3)R_{stsw}{}^4Q_{stsw} + (\beta^4/4)R_{stsw}{}^5]$$

The general equation resulting from this approach is as follows:

$$R_{M+N} = \sum_{k=0}^{\epsilon}\binom{M}{k}Q_{st}{}^kR_{st}{}^{M-k}\sum_{n=0}^{N-k}\binom{n+\alpha-1}{n} \times \beta^n(1-\beta)^\alpha\sum_{x=0}^{N-k-n}\binom{N}{x}Q_{stsw}{}^xR_{stsw}{}^{N-x} \quad (12)$$

or

$$R_{M+N} = \sum_{k=0}^{\epsilon}\binom{M}{k}Q_{st}{}^kR_{st}{}^{M-k}\sum_{m=0}^{N-k}\alpha^m\sum_{n=m}^{N-k}(-1)^{n-m} \times S_n{}^{(m)}\frac{\beta^n}{n!}(1-\beta)^\alpha\sum_{x=0}^{N-k-n}\binom{N}{x}Q_{stsw}{}^xR_{stsw}{}^{N-x} \quad (12a)$$

where ϵ is the smaller of M and N since the first term is related to the number of start failures of active units at the beginning of the mission and this number cannot exceed the smaller of M and N. Also in this equation by definition[2] $\binom{k}{x} = 0$ for $k < x$ and $x > 0$; also $\binom{k}{0} = 1$ for all values of k, including $k = -1$. Equations (12) represent the most general case studied; all previous system equations can be derived from Eqs. (12) by the appropriate simplifying assumptions (and certain algebraic rearrangements, where necessary).

The last summation in Eqs. (12) can also be written in negative binomial form if desired:

$$\sum_{x=0}^{N-k-n}\binom{N}{x}Q_{stsw}{}^xR_{stsw}{}^{N-x} = \sum_{r=0}^{N-k-n}\binom{r+k+n-1}{r}Q_{stsw}{}^rR_{stsw}{}^{n+k}$$

Although the normal binomial form allows a more compact notation, the expansion of terms is simpler and more efficient when the negative binomial form is used. This is illustrated in the following example and is discussed in more detail in the Appendix. To aid in the use of these expansions, Eqs. (12) will be written in expanded form for a "2 + 3" system. From Eq. (12), using the negative binomial form of the third summation,

$$R_{2+3} = R_u{}^2\{R_{st}{}^2[1 + \alpha\beta R_{stsw}(1 + Q_{stsw} + Q_{stsw}{}^2) + (\alpha+1)\alpha(\beta^2/2!)R_{stsw}{}^2(1 + 2Q_{stsw}) + (\alpha+2) \times (\alpha+1)\alpha(\beta^3/3!)R_{stsw}{}^3] + 2Q_{st}R_{st}[R_{stsw}(1 + Q_{stsw} + Q_{stsw}{}^2) + \alpha\beta R_{stsw}{}^2(1 + 2Q_{stsw}) + (\alpha+1)\alpha(\beta^2/2!) \times R_{stsw}{}^3] + Q_{st}{}^2[R_{stsw}{}^2(1 + 2Q_{stsw}) + \alpha\beta R_{stsw}{}^3]\}$$

From Eq. (12a), using the normal binomial form of the third summation and after inserting the numerical values for the Stirling numbers where, by definition $S_n{}^{(0)} = 0$ for $n > 0$ and $S_0{}^{(0)} = 1$,

$$R_{2+3} = R_{st}{}^2R_u{}^2\Big\{[R_{stsw}{}^3 + 3R_{stsw}{}^2Q_{stsw} + 3R_{stsw}Q_{stsw}{}^2 + Q_{stsw}{}^3] + \alpha\Big[\beta(R_{stsw}{}^3 + 3R_{stsw}{}^2Q_{stsw} + 3R_{stsw}Q_{stsw}{}^2) + \frac{\beta^2}{2!}(R_{stsw}{}^3 + 3R_{stsw}{}^2Q_{stsw}) + 2\frac{\beta^3}{3!}R_{stsw}{}^3\Big] + \alpha^2\Big[\frac{\beta^2}{2!}(R_{stsw}{}^3 + 3R_{stsw}{}^2Q_{stsw}) + 3\frac{\beta^3}{3!}R_{stsw}{}^3\Big] + \frac{\alpha^3\beta^3}{3!}R_{stsw}{}^3\Big\} + 2Q_{st}R_{st}R_u{}^2\Big\{[R_{stsw}{}^3 + 3R_{stsw}{}^2Q_{stsw} + 3R_{stsw}Q_{stsw}{}^2] + \alpha\Big[\beta(R_{stsw}{}^3 + 3R_{stsw}{}^2Q_{stsw}) + \frac{\beta^2}{2!}R_{stsw}{}^3\Big] + \frac{\alpha^2\beta^2}{2!}R_{stsw}{}^3\Big\} + Q_{st}{}^2R_u{}^2 \times \{[R_{stsw}{}^3 + 3R_{stsw}{}^2Q_{stsw}] + \alpha\beta R_{stsw}{}^3\}$$

The equivalence of the two expansions is easily demonstrated.

Referring back to Fig. 2, the effect of R_{st} and λ_s/λ_0 on a "2 + 3" system ($R_{sw} = 1.0$) may be seen for three values of $\lambda_0 t$. As shown by the curves for $\lambda_0 t = 0.10$, the difference between the Poisson and binomial reliabilities decreases as R_{st} decreases. This indicates that for $Q_{st} \cong \lambda_0 t$ (and $0 \leq \lambda_s \leq \lambda_0$), the standby units fail more often during start than in the standby mode, regardless of the standby failure rate.

Figure 5 presents R_{2+3} as a function of the start and switching reliabilities for fixed values of $\lambda_0 t$ and λ_s/λ_0. As expected, the start reliability is significantly more important than that of switching since all five units must start whereas only the three redundant units must be switched into the system.

A more practical type of plot is shown in Fig. 6, where the reliability of a "1 + N" system is shown as a function of the redundancy level N. (A plot of this type is actually valid only at integral values of N; the points are connected for clarity.) This type of plot is used, for example, when it is known that one unit must be operating for the mission time, the mean number of operating failures ($\lambda_0 t$) during the mission is 0.20, and it is required to determine the number of redundant units (N) necessary to achieve a specified mission reliability. The six curves illustrate the variation in system reliability for three values of λ_s and two values of R_{st} and R_{sw}. Thus, for a mission reliability (R_{1+N}) goal of 0.998, the number of redundant units required varies from two for $\lambda_s = 0$ with perfect start and switching ($R_{st} = R_{sw} = 1.0$) to five for $\lambda_s = \lambda_0$ with $R_{st} = R_{sw} = 0.9$.

Frequently this type of application also leads to reliability tradeoff studies to arrive at the optimum method of achieving the reliability goal. Such a study is presented in Fig. 7 for a "1 + 3" system—a case where M and N are specified and it

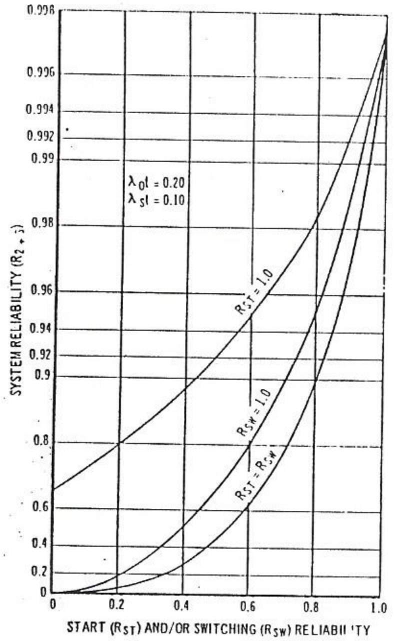

Fig. 5 Effect of start and switching on a 2 + 3 system.

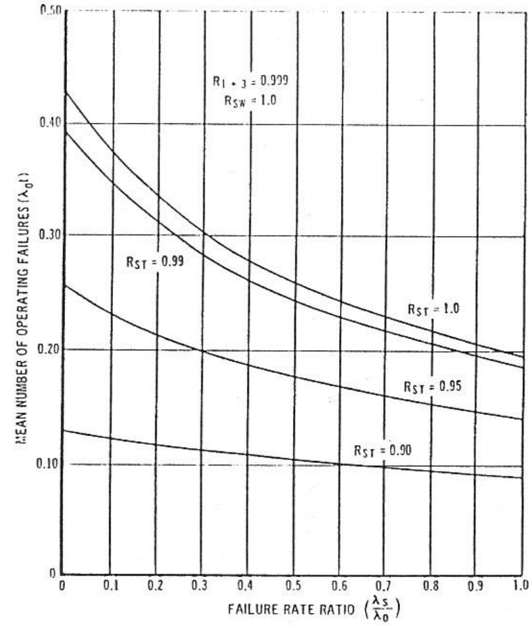

Fig. 7 Parametric analysis for $R_{(1+3)}$ = 0.999.

is desired to examine the relationships between the operating failure rate, the standby failure rate, and the start reliability to obtain the mission reliability of 0.999.

Table 1 presents a summary of the equations developed in this paper. This table is self-explanatory and includes the three modes of redundancy as well as the effects of start and switching reliabilities, both separately and in combined form. The equations presented for $R_{sw} = 1.0$ are obtained from Eq. (12) after an algebraic rearrangement.

It is also of interest to present the system reliability equation in a form that allows definite interpretation of the individual terms. Equation (12a) can be written as

$$R_{M+N} = \sum_{k=0}^{\epsilon} \binom{M}{k} Q_{st}^k R_{st}^{M-k} \sum_{m=0}^{N-k} \frac{(M\lambda_0 t)^m}{m!} \times e^{-M\lambda_0 t} \sum_{n=m}^{N-k} (-1)^{n-m} S_n^{(m)} \frac{m!}{n!} \left[\frac{1 - e^{-\lambda_s t}}{\lambda_s t}\right]^m \times [1 - e^{-\lambda_s t}]^{n-m} \sum_{x=0}^{N-k-n} \binom{N}{x} Q_{stsw}^x R_{stsw}^{N-x}$$

The general term of the foregoing equation represents a certain combination of probabilities:

$$P(M, N, k, m, n, x) = \left[\binom{M}{k} Q_{st}^k R_{st}^{M-k}\right] \times \left[\frac{(M\lambda_0 t)^m}{m!} e^{-M\lambda_0 t}\right]\left[(-1)^{n-m} S_n^{(m)} \frac{m!}{n!} \left(\frac{1 - e^{-\lambda_s t}}{\lambda_s t}\right)^m \times (1 - e^{-\lambda_s t})^{n-m}\right]\left[\binom{N}{x} Q_{stsw}^x R_{stsw}^{N-x}\right]$$

$P(M, N, k, m, n, x)$ represents the probability of success of an "$M + N$" system with exactly k start failures at the beginning of the mission, exactly m random failures in the operating mode (λ_0), exactly $n - m$ random failures in the standby mode (λ_s) with exactly m standby units available as required, and exactly x start or switching failures of the standby units ($x_{max} = N - k - n$).

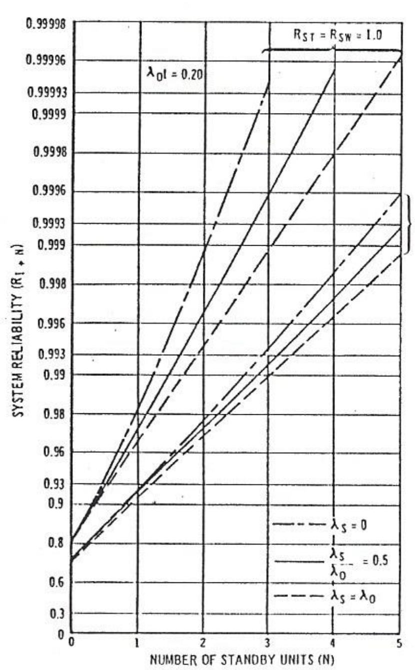

Fig. 6 Effect of redundancy on a 1 + N system.

Conclusions

The conclusions to be drawn from the foregoing presentation can best be summed in relation to the following contributions to reliability engineering:

1) The Poisson-binomial PDF has been developed as the correct method of calculating mission reliability for active-standby redundant systems where all units have equal operating (constant) failure rates and the standby units have a different (constant) failure rate in the standby mode. It is expected that the application of the Poisson-binomial PDF will not be limited to reliability engineering just as that of the Poisson and binomial PDF's is not so limited.

2) The modified Poisson, binomial, negative binomial, and Poisson-binomial PDF's, which include the effects of unit start and switching, have been developed.

3) A clear, concise, and exact method of obtaining complex reliability equations by success path identification, integration of the appropriate probability density functions, and summation of the success path probabilities was presented. This method can be used to develop equations for systems with other failure rate modes; e.g., where the operating failure rates of M and N units are different.

4) The Poisson-binomial PDF which provides additional information and understanding of the Poisson and binomial PDF's, particularly regarding the use of binomial and negative binomial-type, noninteger coefficients, was developed.

Table I Reliability equations for redundant systems[a]

Start and switching effects	Type of redundancy: Active-standby redundancy ($\lambda_s \neq \lambda_0$) or active redundancy ($\lambda_s = \lambda_0$)	Type of redundancy: Standby redundancy ($\lambda_s = 0$) (no standby failures)
General		
Start (R_{st}) and switching (R_{sw}) reliabilities are included. The N redundant units must be started and switched into the sytem; the M initial operating units need only be started.	$R_{M+N} = \sum_{k=0}^{\epsilon} \binom{M}{k} Q_{st}^k R_{st}^{M-k} \sum_{n=0}^{N-k} \binom{n+\alpha-1}{n} \times (1-e^{-\lambda_s t})^n (e^{-M\lambda_0 t}) \sum_{x=0}^{N-k-n} \binom{N}{x} Q_{stsw}^x R_{stsw}^{N-x}$	$R_{M+N} = \sum_{k=0}^{\epsilon} \binom{M}{k} Q_{st}^k R_{st}^{M-k} \sum_{n=0}^{N-k} \frac{(M\lambda_0 t)^n}{n!} \times e^{-M\lambda_0 t} \sum_{x=0}^{N-k-n} \binom{N}{x} Q_{stsw}^x R_{stsw}^{N-x}$
Perfect unit switching ($R_{sw} = 1$) All units (M and N) must be started; switching is assured (or absent).	$R_{M+N} = \sum_{n=0}^{N} \binom{n+\alpha-1}{n} (1-e^{-\lambda_s t})^n (e^{-M\lambda_0 t}) \times \sum_{x=0}^{N-n} \binom{M+N}{x} Q_{st}^x R_{st}^{M+N-x}$	$R_{M+N} = \sum_{n=0}^{N} \frac{(M\lambda_0 t)^n}{n!} e^{-M\lambda_0 t} \times \sum_{x=0}^{N-n} \binom{M+N}{x} Q_{st}^x R_{st}^{M+N-x}$
Perfect unit start ($R_{st} = 1$) Start is assured (or absent). The only degradation in reliability results from switching the redundant units (N) into the system ($Q_{sw} = 1 - R_{sw}$).	$R_{M+N} = \sum_{n=0}^{N} \binom{n+\alpha-1}{n} (1-e^{-\lambda_s t})^n (e^{-M\lambda_0 t}) \times \sum_{x=0}^{N-n} \binom{N}{x} Q_{sw}^x R_{sw}^{N-x}$	$R_{M+N} = \sum_{n=0}^{N} \frac{(M\lambda_0 t)^n}{n!} e^{-M\lambda_0 t} \times \sum_{x=0}^{N-n} \binom{N}{x} Q_{sw}^x R_{sw}^{N-x}$
Perfect unit start and switching ($R_{sw} = R_{st} = 1$) The applicable equations are the standard Poisson and the new Poisson-binomial probability distribution functions.	$R_{M+N} = \sum_{n=0}^{N} \binom{n+\alpha-1}{n} (1-e^{-\lambda_s t})^n (e^{-M\lambda_0 t})$ Poisson-binomial probability distribution function	$R_{M+N} = \sum_{n=0}^{N} \frac{(M\lambda_0 t)^n}{n!} e^{-M\lambda_0 t}$ Poisson probability distribution function

[a] ϵ is the smaller of M and N, $\alpha = M(\lambda_0/\lambda_s)$, $R_{stsw} = R_{st}R_{sw}$, $Q_{stsw} = 1 - R_{stsw}$, and $Q_{st} = 1 - R_{st}$.

Appendix: Negative Binomial PDF

The negative binomial PDF is not commonly used in reliability analysis whereas the binomial PDF is perhaps overused. This is unfortunate since 1) the negative binomial PDF is the most expeditious method of determining the level of redundancy required to meet a specified reliability objective and 2) the negative binomial PDF is equivalent to the binomial PDF for the calculation of mission reliability; in fact, it is simply an algebraic rearrangement of the binomial.[8] It should be noted, however, that the terms of these PDF's sum differently to unity. The binomial PDF consists of a finite population ($M + N$) with the sum of the finite terms being equal to unity; the negative binomial PDF consists of an infinite population (similar to the Poisson PDF) and approaches unity as the number of terms approaches infinity.

The utility of the negative binomial PDF, as contrasted to that of the binomial PDF, may be demonstrated by a practical example. Suppose that for a particular mission 1) at least two units are required; 2) the mission reliability objective (R_{2+N}) is 0.9999; 3) the reliability of a single unit (R_u) is 0.9; and 4) the required number of redundant units, N, is to be determined. The binomial PDF will be considered first:

$$R_{M+N} = R_u^M \sum_{n=0}^{N} \binom{M+N}{n} R_u^{N-n} \cdot Q_u^n$$

The inconvenience of using the binomial PDF for this type of problem stems from the fact that the number of standby units N appears in the binomial coefficients; thus a separate expansion is required for each value of N chosen. For example, if $N = 2$,

$$R_{2+2} = R_u^2 \sum_{n=0}^{2} \binom{4}{n} R_u^{2-n} \cdot Q_u^n = R_u^4 + 4R_u^3 Q_u + 6R_u^2 Q_u^2$$

Each term in the preceding PDF may be identified as follows: R_u^4 is the probability of all four subsystems succeeding even though only two are required (i.e., the probability of exactly zero failures); $4R_u^3 Q_u$ is the probability of any three units succeeding and the last unit failing (i.e., the probability of exactly one failure); $6R_u^2 Q_u^2$ is the probability of any two units succeeding and the remaining two units failing (i.e., the probability of exactly two failures). For $R_u = 0.9$, $R_{2+2} = 0.9963$, which is inadequate. Increasing N to three units results in

$$R_{2+3} = R_u^5 + 5R_u^4 Q_u + 10R_u^3 Q_u^2 + 10R_u^2 Q_u^3 = 0.99954$$

which is still inadequate. Expanding once more for an N of four units yields

$$R_{2+4} = R_u^6 + 6R_u^5 Q_u + 15R_u^4 Q_u^2 + 20R_u^3 Q_u^3 + 15R_u^2 Q_u^4 = 0.999945$$

	COEFFICIENTS	EXAMPLE (M = 5, N = 4, n = 3)
BINOMIAL	$\binom{M+N}{n}$	$\binom{5+4}{3} = \binom{9}{3} = 84$
NEGATIVE BINOMIAL	$\binom{n+M-1}{n}$	$\binom{3+5-1}{3} = \binom{7}{3} = 35$

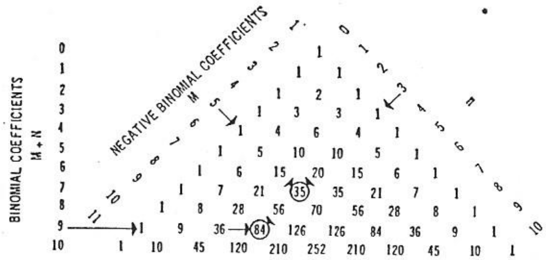

Fig. 8 Binomial coefficients (Pascal's triangle).

which achieves the reliability objective of 0.9999; therefore, six units would be required.

Repeating the analysis with the negative binomial PDF indicates the efficiency of this method for this problem. The negative binomial PDF is

$$R_{M+N} = R_u{}^M \sum_{n=0}^{N} \binom{n + M - 1}{n} Q_u{}^n$$

For $M = 2$,

$$R_{2+N} = R_u{}^2 \sum_{n=0}^{N} \binom{n+1}{n} Q_u{}^n = R_u{}^2 \sum_{n=0}^{N} (n+1)Q_u{}^n$$

$$R_{2+N} = R_u{}^2[1 + 2Q_u + 3Q_u{}^2 + \ldots + (N+1)Q_u{}^N]$$

Thus, only one expansion is required for any number of redundant units; the addition of another redundant unit simply adds another term to the expansion. For example, if $N = 2$,

$$R_{2+2} = R_u{}^2(1 + 2Q_u + 3Q_u{}^2)$$

As in the binomial PDF, each term in the negative binomial may be identified as follows: $R_u{}^2$ is the probability that the first two units will complete the mission. No standby units are required and success or failure of such units is immaterial (i.e., it is the probability of exactly zero failures of active or operating units); $2R_u{}^2Q_u$ is the probability that either one of the first two units will fail to complete the mission and that the first standby unit to be called will complete the mission (i.e., the probability of exactly one failure of an active unit); $3R_u{}^2Q_u{}^2$ is the sum of the probabilities that 1) either one of the first two units will fail, the first standby unit will fail, and the second standby unit will complete the mission; and 2) both of the first two units will fail and the two standby units will complete the mission (i.e., the probability of exactly two failures of either active or standby units). For $N = 3$, $R_{2+3} = R_{2+2} + 4R_u{}^2Q_u{}^3$, etc., until the required reliability objective is achieved. The equivalence of the two expansions can be demonstrated by factoring $R_u{}^2$ from the binomial expansion and replacing the remaining R_u's with $1 - Q_u$; the result will be the negative binomial expansion. This equivalence has been demonstrated for the general case.[8] The coefficients of both the binomial and negative binomial PDF's may also be obtained from Pacsal's triangle, as shown in Fig. 8.

It may be concluded that the negative binomial PDF provides the simplest and most expeditious method for reliability analysis when it is necessary to calculate the number of redundant units necessary to meet a specified mission reliability objective. (A computer program for calculating system reliability as a function of M and N is also simplified by the use of the negative binomial PDF.)

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